Form of cosmic string cusps

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(Received 2 October 1998; published 19 February 1999)

We classify the possible shapes of cosmic string cusps and how they transform under Lorentz boosts. A generic cusp can be brought into a form in which the motion of the cusp tip lies in the plane of the cusp. The cusp whose motion is perpendicular to this plane, considered by some authors, is a special case and not the generic situation. We redo the calculation of the energy in the region where the string overlaps itself near a cusp, which is the maximum energy that can be released in radiation. We take into account the motion of a generic cusp and the resulting Lorentz contraction of the string core. The result is that the energy scales as $\sqrt{rL}$ instead of the usual $r^{1/3}L^{2/3}$, where r is the string radius and L is the typical length scale of the string. Since $r \ll L$ for cosmological strings, the radiation is strongly suppressed and could not be observed. [S0556-2821(99)01406-X]

PACS number(s): 98.80.Cq, 11.27.+d

I. INTRODUCTION

Cosmic strings are topologically stable defects which may have formed during a symmetry breaking phase transition in the early universe. Because of their potential cosmological and astrophysical effects, their properties and evolution have been extensively studied in the past decades. (For reviews see [1,2].) After formation, strings evolve into a scaling network in which a Hubble volume at any time contains of order 1 long string and a larger number of oscillating loops in the process of decay. The decay of cosmic string loops has been studied as a possible source of ultrahigh-energy cosmic rays [3,4].

When the string is smooth on scales of the order of the string thickness, the equations of motion reduce to the Nambu-Goto form, in which the string is treated as a one-dimensional object without thickness. A typical cosmic string loop, in the course of its oscillations, produces one or more cusps: points where, in the Nambu-Goto approximation, the string attains the velocity of light and doubles back on itself. Near a cusp, the string can interact with itself and some of the energy stored in the string can be released as high-energy radiation [5–9]. In this paper we analyze the possible forms of such cusps and discuss how their shape and motion are affected by the choice of Lorentz frame.

We will consider a gauge string with an energy scale η, which leads to a string tension $\mu \sim \eta^2$ and a core radius $r \sim \eta^{-1}$. The first deviation from the Nambu-Goto equations of motion appears at order $(r/R)^4$ [10] where R is the radius of curvature of the string. Far from any cusp (or kink), R is of cosmological size, whereas r is tiny; so the Nambu-Goto approximation is very good. Near the cusp, the two branches of the string can interact with each other, and the Nambu-Goto approximation is no longer accurate. However, any such interaction must fall off rapidly at large separations; so string points whose closest distance of approach is several times r will not be affected by any corrections, and thus will have no chance to interact. Thus the result of the corrections can be, at most, to change the effective radius at which the two branches of the string can be considered to overlap.

Vincent, Antunes, and Hindmarsh [11] found significant departures from Nambu-Goto evolution in a field theory simulation of a string with standing waves, but Moore and Shellard [12] did not find such an effect. We have simulated cusp production in lattice field theory [13] and have not found a significant departure from the Nambu-Goto evolution before the time of the cusp. We will assume here that the amount of energy emission can be calculated by following the Nambu-Goto equations of motion and finding those places where the strings overlap, where overlap is defined using some radius $r \sim \eta^{-1}$. The corrections discussed above might lead, at most, to a change of r by a small numerical factor, but our conclusions would not be affected.

The position of the string at time t can be given by a function $\mathbf{x}(\sigma,t)$. With the usual choice of parametrization (i.e., gauge) the function $\mathbf{x}$ satisfies

$$|\mathbf{x}'(\sigma,t)|^2 + |\dot{\mathbf{x}}(\sigma,t)|^2 = 1, \tag{1a}$$

$$\mathbf{x}'(\sigma,t)\cdot\dot{\mathbf{x}}(\sigma,t) = 0, \tag{1b}$$

and the equation of motion is

$$\mathbf{x}''(\sigma,t) = \ddot{\mathbf{x}}(\sigma,t), \tag{2}$$

where $\mathbf{x}'$ denotes differentiation with respect to σ and $\dot{\mathbf{x}}$ denotes differentiation with respect to t. The general solution is

$$\mathbf{x}(\sigma,t) = \frac{1}{2}[\mathbf{a}(\sigma - t) + \mathbf{b}(\sigma + t)], \tag{3}$$

where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary functions that satisfy $|\mathbf{a}'| = |\mathbf{b}'| = 1$. A cusp is a point at which $\mathbf{a}'(\sigma - t) = -\mathbf{b}'(\sigma + t)$ and thus $\mathbf{x}' = 0$ and $|\dot{\mathbf{x}}| = 1$.

We will expand $\mathbf{x}$, $\mathbf{a}$, and $\mathbf{b}$ in Taylor series around the position of the cusp, which we will take as $\sigma = 0$ and $t = 0$. By dimensional arguments, for a string with a typical length

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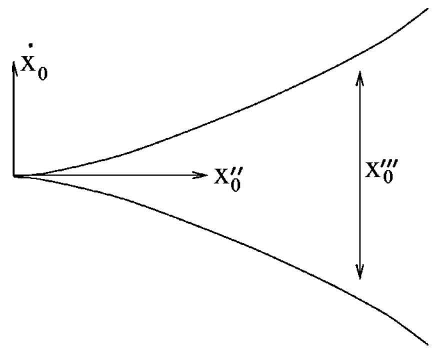

FIG. 1. The parameters of a cusp. The tip moves at the speed of light in the direction $\dot{\mathbf{x}}_0$, the direction of the string near the cusp is given by $\mathbf{x}_0''$, and the spreading of the strings is in the direction $\mathbf{x}_0'''$.

scale L we expect the nth derivatives to be of order L^{1-n}, whereas, as we will see, we expect σ to be of order $\sqrt{rL}$, where r is the string radius. Since r is tiny [only a few orders of magnitude above the Planck scale for a grand unified theory (GUT) scale gauge string], while L is a cosmological size, we see that this expansion converges very rapidly indeed.

We will take the origin of coordinates at the cusp and expand $\mathbf{a}$ and $\mathbf{b}$ near the cusp to third order in σ,

$$\mathbf{a}(\sigma)=\mathbf{a}_0'\sigma+\mathbf{a}_0''\frac{\sigma^2}{2}+\mathbf{a}_0'''\frac{\sigma^3}{6}, \tag{4a}$$

$$\mathbf{b}(\sigma)=\mathbf{b}_0'\sigma+\mathbf{b}_0''\frac{\sigma^2}{2}+\mathbf{b}_0'''\frac{\sigma^3}{6}, \tag{4b}$$

where the subscript 0 denotes quantities at the cusp. To have a cusp we require

$$\mathbf{a}_0'=-\mathbf{b}_0', \tag{5}$$

from which $\dot{\mathbf{x}}_0=\mathbf{b}_0'$ and $\mathbf{x}_0'=0$. Similarly, we can expand

$$\mathbf{x}(\sigma)=\mathbf{x}_0''\frac{\sigma^2}{2}+\mathbf{x}_0'''\frac{\sigma^3}{6}. \tag{6}$$

These vectors are shown in Fig. 1.

To maintain $|\mathbf{a}'(\sigma)|=1$ and $|\mathbf{b}'(\sigma)|=1$ we require

$$\mathbf{a}''\cdot\mathbf{a}'=0, \tag{7a}$$

$$\mathbf{a}'''\cdot\mathbf{a}'=-|\mathbf{a}''|^2, \tag{7b}$$

and similarly for $\mathbf{b}$. From Eqs. (5) and (7) we see that

$$\mathbf{x}_0''\cdot\dot{\mathbf{x}}_0=0, \tag{8a}$$

$$\mathbf{x}_0'''\cdot\dot{\mathbf{x}}_0=\frac{1}{2}(|\mathbf{a}_0''|^2-|\mathbf{b}_0''|^2). \tag{8b}$$

Some authors [5–9] have considered cusps with $\mathbf{x}_0'''\cdot\dot{\mathbf{x}}_0=0$, but in general this will not be the case. We will see later that $|\mathbf{a}_0''|/|\mathbf{b}_0''|$ is unaffected by Lorentz transformations, so that a cusp which does not have $\mathbf{x}_0'''\cdot\dot{\mathbf{x}}_0=0$ cannot be transformed into one which does.

We can align our coordinate system so that $\dot{\mathbf{x}}_0$ lies along the positive x axis. Then $\mathbf{a}_0''$ and $\mathbf{b}_0''$ lie in the y-z plane. Since the overall orientation in this plane is immaterial, the degrees of freedom at the second-derivative level are the magnitudes $|\mathbf{a}_0''|$ and $|\mathbf{b}_0''|$ and the angle between $\mathbf{a}_0''$ and $\mathbf{b}_0''$. Similarly at the third-derivative level, the component of $\mathbf{a}_0'''$ in the x direction is determined by Eq. (7b), and similarly for $\mathbf{b}_0'''$; so we have two degrees of freedom each for $\mathbf{a}_0'''$ and $\mathbf{b}_0'''$. The cusp is specified by seven parameters in all.

II. LORENTZ TRANSFORMATIONS

We now consider the effect of a Lorentz boost on the second- and third-derivative parameters which characterize our cusp. To determine the transformation we proceed as follows. The string world sheet is a two-dimensional surface in four-dimensional space. The surface is timelike except at isolated cusp points, where it is null. A timelike two-surface can be characterized at each point by two forward-directed null vectors A and B. The length of these vectors is arbitrary.

In a particular coordinate system, we can express the position of the string by a function $\mathbf{x}(\sigma,t)$, which we expand as in Eq. (3). The four-vector $(0,\mathbf{x}')$ represents motion along the string at fixed t and thus is tangent to the world sheet. Similarly, the four-vector $(1,\dot{\mathbf{x}})$ represents motion into the future and in the direction of motion of the string; so it also is tangent to the world sheet. The sums and differences of these vectors are null and can be taken as our A and B,

$$A^\mu=(1,\dot{\mathbf{x}}-\mathbf{x}')=(1,-\mathbf{a}'), \tag{9a}$$

$$B^\mu=(1,\dot{\mathbf{x}}+\mathbf{x}')=(1,\mathbf{b}'). \tag{9b}$$

If we now transform to a new coordinate system, the vectors A and B will still be tangent to the world sheet. These vectors will still be null but they will have new components which we will write

$$A^{\tilde{\mu}}=(A^{\tilde{t}},\mathbf{A}), \tag{10a}$$

$$B^{\tilde{\mu}}=(B^{\tilde{t}},\mathbf{B}). \tag{10b}$$

We will denote by $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ the functions $\mathbf{a}$ and $\mathbf{b}$ in the new coordinate system. We can determine $\tilde{\mathbf{a}}'$ and $\tilde{\mathbf{b}}'$ by simply scaling $A^{\tilde{\mu}}$ and $B^{\tilde{\mu}}$,

$$\tilde{A}^{\tilde{\mu}}=(1,-\tilde{\mathbf{a}}')=A/A^{\tilde{t}}, \tag{11a}$$

$$\tilde{B}^{\tilde{\mu}}=(1,\tilde{\mathbf{b}}')=B/B^{\tilde{t}}. \tag{11b}$$

We will let our new coordinate system move with velocity $-\boldsymbol{\beta}$, so that a particle at rest in the original system is moving with velocity $\boldsymbol{\beta}$ with respect to the new coordinates.

The Lorentz transformation then gives $A^{\tilde{t}}=\gamma(1-\mathbf{a}'\cdot\boldsymbol{\beta})$ and $B^{\tilde{t}}=\gamma(1+\mathbf{b}'\cdot\boldsymbol{\beta})$, where $\gamma=1/\sqrt{1-\beta^2}$. We will define

$$f_A=\frac{1}{A^{\tilde{t}}}=\frac{1}{\gamma(1-\mathbf{a}'\cdot\boldsymbol{\beta})}, \tag{12a}$$

$$f_B=\frac{1}{B^{\tilde{t}}}=\frac{1}{\gamma(1+\mathbf{b}'\cdot\boldsymbol{\beta})}. \tag{12b}$$

Now let h be any function defined near the world sheet. We can differentiate in the direction of A or B,

$$A^\mu\partial_\mu h=\left.\frac{\partial h}{\partial t}\right|_{\mathbf{x}}-\mathbf{a}'\cdot\nabla h=\left.\frac{\partial h}{\partial t}\right|_{\mathbf{x}}+\dot{\mathbf{x}}\cdot\nabla h-\mathbf{x}'\cdot\nabla h$$
$$=\left.\frac{\partial h}{\partial t}\right|_{\sigma}-\left.\frac{\partial h}{\partial \sigma}\right|_{t}, \tag{13}$$

and

$$B^\mu\partial_\mu h=\left.\frac{\partial h}{\partial t}\right|_{\mathbf{x}}+\mathbf{b}'\cdot\nabla h=\left.\frac{\partial h}{\partial t}\right|_{\mathbf{x}}+\dot{\mathbf{x}}\cdot\nabla h+\mathbf{x}'\cdot\nabla h$$
$$=\left.\frac{\partial h}{\partial t}\right|_{\sigma}+\left.\frac{\partial h}{\partial \sigma}\right|_{t}. \tag{14}$$

If we make an arbitrary extension of A and B to a neighborhood of the world sheet, we can define

$$B_2^\nu=B^\mu\partial_\mu B^\nu=(0,\dot{\mathbf{b}}'+\mathbf{b}'')=(0,2\mathbf{b}''), \tag{15a}$$

$$A_2^\nu=A^\mu\partial_\mu A^\nu=(0,-\dot{\mathbf{a}}'+\mathbf{a}'')=(0,2\mathbf{a}''). \tag{15b}$$

In the new reference frame we will have $\tilde{A}_2^{\tilde{\mu}}=(0,2\tilde{\mathbf{a}}'')$ where

$$\tilde{A}_2=\tilde{A}^{\tilde{\mu}}\partial_{\tilde{\mu}}\tilde{A}=f_AA^\mu\partial_\mu(f_AA)=f_A^2A^\mu\partial_\mu A+f_Af_{A,A}A$$
$$=f_A^2A_2+f_Af_{A,A}A, \tag{16}$$

with $f_{A,A}=A^\mu\partial_\mu f_A$, and similarly $\tilde{B}_2^{\tilde{\mu}}=(0,2\tilde{\mathbf{b}}'')$ with

$$\tilde{B}_2=\tilde{B}^{\tilde{\mu}}\partial_{\tilde{\mu}}\tilde{B}=f_BB^\mu\partial_\mu(f_BB)=f_B^2B^\mu\partial_\mu B+f_Bf_{B,B}B$$
$$=f_B^2B_2+f_Bf_{B,B}B \tag{17}$$

and $f_{B,B}=B^\mu\partial_\mu f_B$.

We are not concerned with the actual directions of $\mathbf{a}',\mathbf{a}''$, and so on, but only their lengths and the angles between them. We can compute

$$4|\tilde{\mathbf{a}}''|^2=g(\tilde{A},\tilde{A})=f_A^2f_{A,A}^2g(A,A)+2f_A^3f_{A,A}g(A,A_2)$$
$$+f_A^4g(A_2,A_2), \tag{18}$$

where we have used a metric $g=\mathrm{diag}(-1,1,1,1)$.

Since A is null, $g(A,A)=0$. We also have $g(A,A_2)=2\mathbf{a}'\cdot\mathbf{a}''=0$ from Eq. (7a) and so

$$|\tilde{\mathbf{a}}''|=f_A^2|\mathbf{a}''|. \tag{19}$$

Similarly, $|\tilde{\mathbf{b}}''|=f_B^2|\mathbf{b}''|$.

At the cusp, $A=B$ and so we can let

$$f=f_A=f_B. \tag{20}$$

At this point $g(A,B)=0$ and $g(A_2,B)=g(B_2,A)=0$; so

$$\tilde{\mathbf{a}}_0''\cdot\tilde{\mathbf{b}}_0''=f^4\mathbf{a}_0''\cdot\mathbf{b}_0''. \tag{21}$$

This means that the angle between $\mathbf{a}_0''$ and $\mathbf{b}_0''$ is unaffected by the boost. At the second-derivative level, the effect of a Lorentz transformation (up to rotation) is purely to rescale $\mathbf{a}_0''$ and $\mathbf{b}_0''$ by the same factor f^2.

We can compute the third derivatives using the same technique,

$$(0,-4\mathbf{a}''')=A_3^\rho=A^\mu\partial_\mu(A^\nu\partial_\nu A^\rho), \tag{22a}$$

$$(0,4\mathbf{b}''')=B_3^\rho=B^\mu\partial_\mu(B^\nu\partial_\nu B^\rho) \tag{22b}$$

and

$$\tilde{A}_3=f_AA^\mu\partial_\mu(f_AA^\nu\partial_\nu(f_AA))$$
$$=f_A^3A_3+3f_A^2f_{A,A}A_2+(f_Af_{A,A}^2+f_A^2f_{A,AA})A, \tag{23a}$$

$$\tilde{B}_3=f_BB^\mu\partial_\mu(f_BB^\nu\partial_\nu(f_BB))$$
$$=f_B^3B_3+3f_B^2f_{B,B}B_2+(f_Bf_{B,B}^2+f_B^2f_{B,BB})B, \tag{23b}$$

where $f_{A,AA}=A^\mu\partial_\mu(A^\nu\partial_\nu f_A)$ and $f_{B,BB}=B^\mu\partial_\mu(B^\nu\partial_\nu f_B)$.

The component of $\mathbf{a}_0'''$ in the direction of $\dot{\mathbf{x}}_0$ is fixed by Eq. (7b); so we are only interested in the components in the directions of $\mathbf{a}_0''$ and $\mathbf{b}_0''$. When we take the inner product of Eqs. (23a) and (16), only the first and second terms of Eq. (23a) will contribute as before:

$$-8\tilde{\mathbf{a}}'''\cdot\tilde{\mathbf{a}}''=g(\tilde{A}_3,\tilde{A}_2)$$
$$=f_A^5g(A_3,A_2)+f_A^4f_{A,A}g(A_3,A)$$
$$+3f_A^4f_{A,A}g(A_2,A_2)$$
$$=-8f_A^5\mathbf{a}'''\cdot\mathbf{a}''+4f_A^4f_{A,A}\mathbf{a}'''\cdot\mathbf{a}'+12f_A^4f_{A,A}|\mathbf{a}''|^2. \tag{24}$$

Using Eq. (7b) we can write this

$$\tilde{\mathbf{a}}'''\cdot\tilde{\mathbf{a}}''=f_A^5\mathbf{a}'''\cdot\mathbf{a}''-f_A^4f_{A,A}|\mathbf{a}''|^2. \tag{25}$$

Now, from Eqs. (12a) and (13),

$$f_{A,A}=\left.\frac{\partial f_A}{\partial t}\right|_{\sigma}-\left.\frac{\partial f_A}{\partial \sigma}\right|_{t}=\frac{\boldsymbol{\beta}}{\gamma(1-\mathbf{a}'\cdot\boldsymbol{\beta})^2}\cdot(\dot{\mathbf{a}}'-\mathbf{a}'') \tag{26}$$

$$=-f_A\frac{2\boldsymbol{\beta}\cdot\mathbf{a}''}{1-\mathbf{a}'\cdot\boldsymbol{\beta}}=-2\gamma f_A^2\boldsymbol{\beta}\cdot\mathbf{a}'' \tag{27}$$

and, similarly,

$$f_{B,B}=\frac{\partial f_B}{\partial t}\bigg|_\sigma+\frac{\partial f_B}{\partial\sigma}\bigg|_t=\frac{-\boldsymbol{\beta}}{\gamma(1+\mathbf{b}'\cdot\boldsymbol{\beta})^2}\cdot(\dot{\mathbf{b}}'+\mathbf{b}'') \tag{28}$$

$$=-f_B\frac{2\boldsymbol{\beta}\cdot\mathbf{b}''}{1+\mathbf{b}'\cdot\boldsymbol{\beta}}=-2\gamma f_B^2\boldsymbol{\beta}\cdot\mathbf{b}''. \tag{29}$$

Equation (25) then becomes

$$\tilde{\mathbf{a}}'''\cdot\tilde{\mathbf{a}}''=f_A^5(\mathbf{a}'''\cdot\mathbf{a}''+2\gamma f_A|\mathbf{a}''|^2\boldsymbol{\beta}\cdot\mathbf{a}''). \tag{30}$$

In the same manner we can compute

$$\tilde{\mathbf{b}}'''\cdot\tilde{\mathbf{b}}''=f_B^5(\mathbf{b}'''\cdot\mathbf{b}''-2\gamma f_B|\mathbf{b}''|^2\boldsymbol{\beta}\cdot\mathbf{b}''), \tag{31}$$

and, at the cusp, using Eq. (20),

$$\tilde{\mathbf{a}}_0'''\cdot\tilde{\mathbf{b}}_0''=f^5[\mathbf{a}_0'''\cdot\mathbf{b}_0''+\gamma f(3(\mathbf{a}_0''\cdot\mathbf{b}_0'')\boldsymbol{\beta}\cdot\mathbf{a}_0''-|\mathbf{a}_0''|^2\boldsymbol{\beta}\cdot\mathbf{b}_0'')], \tag{32a}$$

$$\tilde{\mathbf{b}}_0'''\cdot\tilde{\mathbf{a}}_0''=f^5[\mathbf{b}_0'''\cdot\mathbf{a}_0''+\gamma f(|\mathbf{b}_0''|^2\boldsymbol{\beta}\cdot\mathbf{a}_0''-3(\mathbf{a}_0''\cdot\mathbf{b}_0'')\boldsymbol{\beta}\cdot\mathbf{b}_0'')]. \tag{32b}$$

Now we consider two types of Lorentz transformation: a longitudinal boost in which $\boldsymbol{\beta}$ is parallel to $\dot{\mathbf{x}}_0$, and a boost with a transverse component but which has $f=1$. For the longitudinal boost, the directions of $\dot{\mathbf{x}}_0$, $\mathbf{a}_0''$, and $\mathbf{b}_0''$ are unaffected. For the $f=1$ boost we will rotate the system after the boost so that these vectors are returned to their original directions.

For a longitudinal boost with $\boldsymbol{\beta}=\beta\dot{\mathbf{x}}_0$, the effect is to rescale the various parameters,

$$\tilde{\mathbf{a}}_0''=f^2\mathbf{a}_0'', \tag{33a}$$

$$\tilde{\mathbf{b}}_0''=f^2\mathbf{b}_0'', \tag{33b}$$

$$\tilde{\mathbf{a}}_0'''^{(\perp)}=f^5\mathbf{a}_0'''^{(\perp)}, \tag{33c}$$

$$\tilde{\mathbf{b}}_0'''^{(\perp)}=f^5\mathbf{b}_0'''^{(\perp)}, \tag{33d}$$

$$\tilde{\mathbf{a}}_0'''^{(\parallel)}=f^4\mathbf{a}_0'''^{(\parallel)}, \tag{33e}$$

$$\tilde{\mathbf{b}}_0'''^{(\parallel)}=f^4\mathbf{b}_0'''^{(\parallel)}, \tag{33f}$$

where $(\perp)$ denotes components perpendicular to $\dot{\mathbf{x}}_0$ and $(\parallel)$ denotes parallel components, and

$$f=\frac{1}{\gamma(1+\beta)}=\frac{\sqrt{1-\beta^2}}{1+\beta}=\sqrt{\frac{1-\beta}{1+\beta}}. \tag{34}$$

We see that $1/f$ is just the Doppler shift factor for radiation moving in the $\dot{\mathbf{x}}_0$ direction.

For an $f=1$ transformation, $\mathbf{a}_0'$, $\mathbf{b}_0'$, $\mathbf{a}_0''^{(\parallel)}$, and $\mathbf{b}_0''^{(\parallel)}$ are unchanged, and

$$\tilde{\mathbf{a}}_0'''\cdot\tilde{\mathbf{a}}_0''=(\mathbf{a}_0'''\cdot\mathbf{a}_0''+2\gamma|\mathbf{a}_0''|^2\boldsymbol{\beta}\cdot\mathbf{a}_0''), \tag{35a}$$

$$\tilde{\mathbf{a}}_0'''\cdot\tilde{\mathbf{b}}_0''=\mathbf{a}_0'''\cdot\mathbf{b}_0''+\gamma(3(\mathbf{a}_0''\cdot\mathbf{b}_0'')\boldsymbol{\beta}\cdot\mathbf{a}_0''-|\mathbf{a}_0''|^2\boldsymbol{\beta}\cdot\mathbf{b}_0''), \tag{35b}$$

$$\tilde{\mathbf{b}}_0'''\cdot\tilde{\mathbf{a}}_0''=\mathbf{b}_0'''\cdot\mathbf{a}_0''+\gamma(|\mathbf{b}_0''|^2\boldsymbol{\beta}\cdot\mathbf{a}_0''-3(\mathbf{a}_0''\cdot\mathbf{b}_0'')\boldsymbol{\beta}\cdot\mathbf{b}_0''), \tag{35c}$$

$$\tilde{\mathbf{b}}_0'''\cdot\tilde{\mathbf{b}}_0''=(\mathbf{b}_0'''\cdot\mathbf{b}_0''-2\gamma|\mathbf{b}_0''|^2\boldsymbol{\beta}\cdot\mathbf{b}_0''). \tag{35d}$$

There are two parameters which specify the $f=1$ transformation. For example, we can write

$$\boldsymbol{\beta}=\beta_\parallel\dot{\mathbf{x}}_0+\boldsymbol{\beta}_\perp, \tag{36}$$

where $\boldsymbol{\beta}_\perp$ is perpendicular to $\dot{\mathbf{x}}_0$. To make $f=1$ we demand that $\gamma(1-\beta_\parallel)=1$ or $\sqrt{1-\beta^2}=1-\beta_\parallel$; so $|\boldsymbol{\beta}_\perp|^2=2\beta_\parallel-2\beta_\parallel^2$. We can specify $\beta_\parallel$ and also the direction of $\boldsymbol{\beta}_\perp$ in the plane perpendicular to $\dot{\mathbf{x}}_0$.

By the use of these two degrees of freedom, we can impose two constraints on $\mathbf{a}_0'''$ and $\mathbf{b}_0'''$. For example, we can require that $\mathbf{x}_0'''$ be parallel to $\dot{\mathbf{x}}_0$, as follows. We seek an $f=1$ transformation such that

$$0=\tilde{\mathbf{x}}_0'''\cdot\tilde{\mathbf{a}}_0''=\mathbf{x}_0'''\cdot\mathbf{a}_0''+\gamma\boldsymbol{\beta}\cdot\left[\left(|\mathbf{a}_0''|^2+\frac{1}{2}|\mathbf{b}_0''|^2\right)\mathbf{a}_0''-\frac{3}{2}(\mathbf{a}_0''\cdot\mathbf{b}_0'')\mathbf{b}_0''\right], \tag{37a}$$

$$0=\tilde{\mathbf{x}}_0'''\cdot\tilde{\mathbf{b}}_0''=\mathbf{x}_0'''\cdot\mathbf{b}_0''+\gamma\boldsymbol{\beta}\cdot\left[\frac{3}{2}(\mathbf{a}_0''\cdot\mathbf{b}_0'')\mathbf{a}_0''-\left(|\mathbf{b}_0''|^2+\frac{1}{2}|\mathbf{a}_0''|^2\right)\mathbf{b}_0''\right]. \tag{37b}$$

These equations can be written in the form

$$\gamma\boldsymbol{\beta}\cdot\boldsymbol{\alpha}_1=c_1, \tag{38a}$$

$$\gamma\boldsymbol{\beta}\cdot\boldsymbol{\alpha}_2=c_2, \tag{38b}$$

where $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are vectors in the plane of $\mathbf{a}_0''$ and $\mathbf{b}_0''$, and c_1 and c_2 are constants. Assuming that the vectors $\mathbf{a}_0''$ and $\mathbf{b}_0''$ are linearly independent, $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ will be also, since one can show that the transformation matrix is not singular. Thus we can choose a direction $\hat{\boldsymbol{\beta}}_\perp$ such that

$$\frac{\hat{\boldsymbol{\beta}}_\perp\cdot\boldsymbol{\alpha}_1}{c_1}=\frac{\hat{\boldsymbol{\beta}}_\perp\cdot\boldsymbol{\alpha}_2}{c_2}\equiv c>0. \tag{39}$$

Then we must find a value for $\beta_\parallel$ such that

$$\gamma|\boldsymbol{\beta}_\perp|=1/c. \tag{40}$$

Since $f=1,\gamma=1/(1-\beta_\parallel)$ and

$$\gamma|\boldsymbol{\beta}_\perp| = \frac{|\boldsymbol{\beta}_\perp|}{1-\beta_\parallel} = \sqrt{\frac{2\beta_\parallel}{1-\beta_\parallel}}, \tag{41}$$

which can take any value by appropriate choice of $\beta_\parallel$. Thus by appropriate choice of parameters we can make $\mathbf{x}_0'''\cdot\mathbf{a}_0'' = \mathbf{x}_0'''\cdot\mathbf{b}_0''=0$ so that only the component of $\mathbf{x}_0'''$ parallel to $\dot{\mathbf{x}}_0$ survives.

If this cusp is part of a string with a typical length scale L, then we expect $\mathbf{a}_0'',\mathbf{b}_0''\sim L^{-1}$ and $\mathbf{x}_0'''\sim L^{-2}$. Thus $|\boldsymbol{\alpha}_1|,|\boldsymbol{\alpha}_2| \sim L^{-3}$ and $c_1,c_2\sim L^{-3}$; so $\gamma|\beta|$ needs only to be of order 1. We do not need a huge boost to go to the frame where $\mathbf{x}_0'''$ is parallel to $\dot{\mathbf{x}}_0$.

Once in this canonical frame, we can make longitudinal boosts to scale $|\mathbf{a}_0''|$ and $|\mathbf{b}_0''|$. Thus, up to boosts, the cusp is given by four parameters: the relative magnitude of $\mathbf{a}_0''$ and $\mathbf{b}_0''$, the angle between them, and two parameters giving $\dot{\mathbf{x}}_0''$ in the plane perpendicular to $\dot{\mathbf{x}}_0$.

From these transformations we can determine the way in which the amount of radiation produced by a cusp scales with the typical length scale L. We will imagine that we know nothing about the process by which this radiation is produced, except for the following assumptions: the radiation is strongly relativistic; the radiation is strongly beamed in the direction of $\dot{\mathbf{x}}_0$; and the amount of radiation depends only on $\mathbf{x}_0''$, $\dot{\mathbf{x}}_0'$, and $\mathbf{x}_0'''$.

Consider a cusp with length scale L and suppose that it emits an amount of radiation E. Now suppose first that we rescale this cusp by a factor s. We will have

$$\mathbf{x}_0'',\dot{\mathbf{x}}_0'\sim s^{-1}, \tag{42a}$$

$$\mathbf{x}_0'''\sim s^{-2}. \tag{42b}$$

Now suppose instead that we transform the cusp by a longitudinal boost with $f=s^{-1/2}$. From Eqs. (33) the parameters scale as

$$\mathbf{x}_0'',\dot{\mathbf{x}}_0'\sim f^2\sim s^{-1}, \tag{43a}$$

$$\mathbf{x}_0'''\sim f^4\sim s^{-2}, \tag{43b}$$

just as in Eqs. (42). Since the radiation is strongly relativistic and strongly beamed in the forward direction, it will transform under the boost as

$$E\rightarrow\sqrt{\frac{1+\beta}{1-\beta}}\,E=\frac{E}{f}=s^{1/2}E. \tag{44}$$

Since the relevant parameters in the rescaled string are the same as in the boosted string, the rescaled string will also release energy $s^{1/2}E$.

III. OVERLAP CALCULATION

Outside the string core, the fields fall off rapidly to their vacuum values. Therefore, the portion of string near the cusp which can be released as radiation can be at most the part where the string cores overlap. We will let r denote the radius of the core at which this annihilation becomes possible. It is not clear precisely what this value should be, but it is approximately the length corresponding to the energy scale of the string, $r\sim10^{-30}$ cm for a GUT-scale gauge string.

We will use the canonical reference frame discussed earlier, in which $\dot{\mathbf{x}}_0$ is parallel to $\mathbf{x}_0'''$.[1] We show in the Appendix that, in this frame, the length of the string overlap is maximized at the time of the cusp ($t=0$). At the order of approximation we will use, the cusp is symmetrical; so we will consider only the overlap between the point of the string at σ and its symmetrical point at $-\sigma$.

Using the Taylor expansion, Eq. (6), of the string around the position of the cusp, the distance between two points on the two different branches of the string is given by

$$|\mathbf{x}(\sigma,0)-\mathbf{x}(-\sigma,0)|=|\mathbf{x}_0'''|\frac{\sigma^3}{3}+\cdots. \tag{45}$$

Using Eq. (1a) we can write the lowest-order correction to the velocity at a particular value of the parameter σ,

$$|\dot{\mathbf{x}}(\sigma,0)|^2=1-|\mathbf{x}'(\sigma,0)|^2. \tag{46}$$

We can now use Eq. (6) to expand $\mathbf{x}'$,

$$\mathbf{x}'(\sigma,0)=\mathbf{x}_0''\sigma+\mathbf{x}_0'''\frac{\sigma^2}{2}+\cdots, \tag{47}$$

so that at the lowest order we have

$$|\dot{\mathbf{x}}(\sigma,0)|^2\simeq1-\sigma^2|\mathbf{x}_0''|^2. \tag{48}$$

The gamma factor at this order in σ is

$$\gamma=\frac{1}{\sqrt{1-\dot{\mathbf{x}}^2}}\simeq\frac{1}{|\mathbf{x}_0''|\sigma}. \tag{49}$$

As a result of Lorentz contraction of the core of the string in the direction of motion, the cross section of the string in this frame is not a circle of radius r, but an ellipse with semimajor axis r and semiminor axis r/γ in the direction of $\dot{\mathbf{x}}$.

Now the overlap calculation reduces to the computation of the value of σ at which these two ellipses touch one another. As before we will only take into account effects of order σ. This allows us to ignore the component of the ve-

[1] If we start with a cusp in which $\mathbf{x}_0'''$ is perpendicular to $\dot{\mathbf{x}}_0$, in the canonical frame we have $\mathbf{x}_0'''=0$. Thus to third order in σ the two parts of the string lie on top of each other. Only at fifth order are they separate; so the radiation emitted will be much larger than we calculate here. If one analyzes this cusp in the original frame, one finds that, at times either before or after the time of the cusp, the string nearly crosses itself. These near crossings give rise to the large energy emission. Since the typical cusp does not have the parameters chosen to produce this exact alignment, this large amount of radiation occurs in only a vanishingly small fraction of cases.

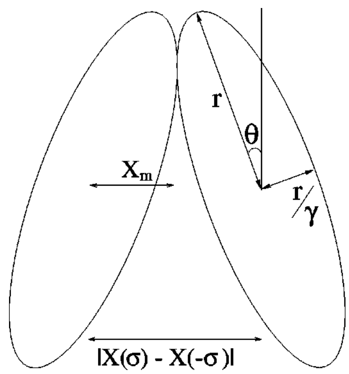

FIG. 2. A cross section through the strings at the point of contact.

locity in the direction parallel to $\mathbf{x}_0''$. The component of $\dot{\mathbf{x}}$ in the plane perpendicular to $\mathbf{x}_0''$ is

$$\dot{\mathbf{x}}_\perp = \dot{\mathbf{x}} - \frac{(\dot{\mathbf{x}}\cdot\mathbf{x}_0'')\mathbf{x}_0''}{|\mathbf{x}_0''|^2}. \tag{50}$$

Since $\dot{\mathbf{x}}_0 \cdot \mathbf{x}_0'' = 0$, we see that

$$\dot{\mathbf{x}}_\perp = \dot{\mathbf{x}}_0 + \sigma \mathbf{x}_3, \tag{51}$$

where $\mathbf{x}_3$ is the component of $\dot{\mathbf{x}}_0'$ perpendicular to $\mathbf{x}_0''$,

$$\mathbf{x}_3 = \dot{\mathbf{x}}_0' - \left[\frac{(\dot{\mathbf{x}}_0'\cdot\mathbf{x}_0'')}{|\mathbf{x}_0''|^2}\right]\mathbf{x}_0''. \tag{52}$$

At this order of approximation the angle between $\dot{\mathbf{x}}_\perp$ and $\dot{\mathbf{x}}_0$ is

$$\theta \simeq \frac{|\mathbf{x}_3|}{|\dot{\mathbf{x}}_\perp|}\sigma. \tag{53}$$

The cross section of the string in the plane perpendicular to $\mathbf{x}_0''$ can be approximated by two ellipses whose centers are separated a distance $|\mathbf{x}(\sigma) - \mathbf{x}(-\sigma)|$ and whose major axes are tilted towards each other, each by a small angle θ. See Fig. 2.

It can be shown that if an ellipse whose major axis is along the y direction is tilted by a positive angle α, the maximum value of x that the new rotated ellipse reaches is

$$x_m = \sqrt{d_1^2 \sin^2\alpha + d_2^2 \cos^2\alpha}, \tag{54}$$

where d_1 and d_2 are the lengths of the semimajor and semiminor axes, respectively. In our case,

$$x_m = \sqrt{r^2 \sin^2\theta + \frac{r^2}{\gamma^2}\cos^2\theta}$$
$$\simeq r\sqrt{\theta^2 + \frac{1}{\gamma^2}} \simeq r\sigma\sqrt{|\mathbf{x}_3|^2 + |\mathbf{x}_0''|^2}. \tag{55}$$

The maximum value of σ at which the two branches of the string touch is given by

$$\frac{\sigma_c^3 |\mathbf{x}_0'''|}{3} = 2x_m, \tag{56}$$

and so

$$\sigma_c = \left(\frac{6r\sqrt{|\mathbf{x}_3|^2 + |\mathbf{x}_0''|^2}}{|\mathbf{x}_0'''|}\right)^{1/2}$$
$$= \left(\frac{6r}{|\mathbf{x}_0'''|}\sqrt{|\dot{\mathbf{x}}_0'|^2 - \frac{(\dot{\mathbf{x}}_0'\cdot\mathbf{x}_0'')^2}{|\mathbf{x}_0''|^2} + |\mathbf{x}_0''|^2}\right)^{1/2}. \tag{57}$$

Since $\mathbf{x}_0'''$ is parallel to $\dot{\mathbf{x}}_0$, we can use Eq. (8b),

$$|\mathbf{x}_0'''| = |\mathbf{x}_0'''\cdot\dot{\mathbf{x}}_0| = \frac{1}{2}\left||\mathbf{a}_0''|^2 - |\mathbf{b}_0''|^2\right| = 2|\dot{\mathbf{x}}_0'\cdot\mathbf{x}_0''|, \tag{58}$$

so that we can write the expression for σ_c as

$$\sigma_c = \left(3r\sqrt{\frac{|\mathbf{x}_0''|^2 + |\dot{\mathbf{x}}_0'|^2}{|\dot{\mathbf{x}}_0'\cdot\mathbf{x}_0''|^2} - \frac{1}{|\mathbf{x}_0''|^2}}\right)^{1/2} \tag{59}$$

or in terms of the derivatives of the vectors **a** and **b** as

$$\sigma_c = \left(6r\sqrt{\frac{2(|\mathbf{a}_0''|^2 + |\mathbf{b}_0''|^2)}{(|\mathbf{b}_0''|^2 - |\mathbf{a}_0''|^2)^2} - \frac{1}{(\mathbf{a}_0'' + \mathbf{b}_0'')^2}}\right)^{1/2}. \tag{60}$$

As discussed earlier, we expect the nth derivatives to be of the order of L^{1-n}; so we finally get that

$$\sigma_c \sim \sqrt{rL} + O(r). \tag{61}$$

The segment of string whose energy could be radiated runs from $-\sigma_c$ to σ_c; so the maximum energy release is

$$E = 2\mu\sigma_c \sim \mu\sqrt{rL}, \tag{62}$$

where μ is the energy per unit length of the string.

Using Eqs. (49) and (61) we can compute the Lorentz boost of the string at the point where the overlap begins,

$$\gamma_c \sim \frac{1}{|\mathbf{x}_0''|\sigma_c} \sim \sqrt{\frac{L}{r}}. \tag{63}$$

For a cosmological string, $L \gg r$ and so $\gamma_c \gg 1$. In fact this boost will normally be so large that the emitted particles will have energies much larger than the Planck scale.

We imagine that the radiation consists of ''X bosons'' with rest mass $m_X \sim \sqrt{\mu}$, so that the number of particles emitted is

$$N_X \sim \frac{E}{\gamma m_X} \sim r\sqrt{\mu} \sim 1, \tag{64}$$

independent of the cosmological length scale L. The physical size of the region from which these particles are emitted is

$$l \sim |\mathbf{x}_0''|\sigma_c^2 \sim r, \tag{65}$$

also independent of L. Thus the radiation from a cusp consists of a small number of particles with very large boosts emitted from a region whose size is similar to the string thickness.

IV. DISCUSSION

We have analyzed the transformation properties of cosmic string cusps under Lorentz boosts. A generic cusp can be brought into a canonical form in which the motion of the cusp tip lies in the plane of the cusp. In general a boost of large magnitude is not needed to go to this frame. Once in such a frame, the cusp can be further boosted to scale one parameter, e.g., $\mathbf{x}''$, to any desired value. All the second-derivative parameters scale together under this transformation. Thus the difference between a cusp that one would expect to find in a large loop and one that one would expect in a small loop is essentially a matter of boosting.

The maximum amount of radiation which can be emitted from a cusp is given by the energy stored in the parts of the string whose cores overlap. Taking into account the Lorentz contraction of the core due to the rapid motion of the string near the cusp, we have found that the emitted energy is at most of order $\mu\sqrt{rL}$. Previous analyses [5–9] have used the result $\mu r^{1/3}L^{2/3}$; the present result differs by a factor $(r/L)^{1/6}$. Since r is of microphysical size, while L is cosmological, the energy emitted is reduced by many orders of magnitude by this effect. Since even neglecting Lorentz contraction the radiation from cusps would at most be barely detectable [6–9], this effect prevents any such observation.

ACKNOWLEDGMENTS

We would like to thank Xavier Siemens and Alex Vilenkin for helpful conversations. This work was supported in part by funding provided by the National Science Foundation. J.J.B.P. is supported in part by the Fundación Pedro Barrie de la Maza.

APPENDIX: TIME VARIATION

The computation of the value of σ at which the two branches of the string overlap involves three different effects which have different time dependence; so there is the possibility that portions of the string might overlap at times $t>0$ or $t<0$ that are separate at $t=0$. In order to clarify this point, we redo the calculation, keeping the dependence on time up to second order. Taking into account that in this frame $\mathbf{x}_0'''\cdot\dot{\mathbf{x}}_0'=0$, the distance between the string centers is now

$$|\mathbf{x}(\sigma,t)-\mathbf{x}(-\sigma,t)|^2 \simeq \frac{\sigma^6}{9}|\mathbf{x}_0'''|^2+4t^2\sigma^2|\dot{\mathbf{x}}_0'|^2+\cdots, \tag{A1}$$

and the Lorentz factor is

$$\gamma=\frac{1}{\sqrt{1-\dot{\mathbf{x}}^2}}=\frac{1}{|\mathbf{x}'|}, \tag{A2}$$

and so

$$\frac{1}{\gamma^2}\simeq|\mathbf{x}_0''|^2\sigma^2+2(\mathbf{x}_0''\cdot\dot{\mathbf{x}}_0')\sigma t+|\dot{\mathbf{x}}_0'|^2t^2\cdots \tag{A3}$$

and the angle of the ellipses is

$$\theta^2\simeq|\dot{\mathbf{x}}_\perp-\dot{\mathbf{x}}_0'|^2\simeq\left(|\dot{\mathbf{x}}_0'|^2-\frac{|\dot{\mathbf{x}}_0'\cdot\mathbf{x}_0''|^2}{|\mathbf{x}_0''|^2}\right)\sigma^2+O(\sigma t^2)+\cdots. \tag{A4}$$

Using Eq. (55) we can compute the radius of the string core which would make the two branches of the string touch at σ,

$$r(\sigma,t)^2=\frac{|\mathbf{x}(\sigma,t)-\mathbf{x}(-\sigma,t)|^2}{\theta^2+1/\gamma^2}, \tag{A5}$$

which can be written in terms of the dependence on σ and t as

$$r(\sigma,t)^2=\frac{O(\sigma^6)+O(t^2\sigma^2)}{O(\sigma^2)+O(\sigma t)+O(t^2)}. \tag{A6}$$

We now expand this function around $t=0$,

$$r(\sigma,t)^2=r(\sigma,0)^2+\left(\frac{dr^2}{dt}\right)_0 t+\left(\frac{d^2r^2}{dt^2}\right)_0\frac{t^2}{2}+\cdots, \tag{A7}$$

with

$$r(\sigma,0)^2=O(\sigma^4), \tag{A8a}$$

$$\left(\frac{dr^2}{dt}\right)_0=O(\sigma^3), \tag{A8b}$$

$$\left(\frac{d^2r^2}{dt^2}\right)_0=O(1). \tag{A8c}$$

It can also be shown from Eqs. (A1), (A3), and (A4) that $(d^2r^2/dt^2)_0$ is positive; so we can obtain from Eqs. (A8) the time at which $r(\sigma,t)$ is minimum, namely,

$$t_{min}=O(\sigma^3). \tag{A9}$$

If we use this order of σ in Eq. (A6), we see that the minimum value of r^2 is, at our order of approximation, the same as $r(\sigma,0)^2$; so we can compute the value of σ_c using $r(\sigma_c,0)=r$, as above.

[1] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and other Topological Defects* (Cambridge University Press, Cambridge, England, 1994).
[2] M. B. Hindmarsh and T. W. B. Kibble, Rep. Prog. Phys. **58**, 477 (1995).
[3] V. Berezinsky, P. Blasi, and A. Vilenkin, Phys. Rev. D **58**, 103515 (1998).
[4] P. Bhattacharjee and G. Sigl, astro-ph/9811011 (unpublished).
[5] R. H. Brandenberger, Nucl. Phys. **B293**, 812 (1987).
[6] P. Bhattacharjee, Phys. Rev. D **40**, 3968 (1989).
[7] J. H. MacGibbon and R. H. Brandenberger, Nucl. Phys. **B331**, 153 (1990).
[8] J. H. MacGibbon and R. H. Brandenberger, Phys. Rev. D **47**, 2283 (1993).
[9] R. H. Brandenberger, A. T. Sornborger, and M. Trodden, Phys. Rev. D **48**, 940 (1993).
[10] M. Anderson, F. Bonjour, R. Gregory, and J. Stewart, Phys. Rev. D **56**, 8014 (1997).
[11] G. Vincent, N. D. Antunes, and M. Hindmarsh, Phys. Rev. Lett. **80**, 2277 (1998).
[12] J. N. Moore and E. P. S. Shellard, hep-ph/9808336 (unpublished).
[13] J. J. Blanco-Pillado and K. D. Olum (in preparation).